The license plate limitation problem

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Inspiration for this problem comes from this following article:
`https://www.nytimes.com/2008/06/21/world/asia/21china.html`

In order to combat pollution, climate change, and traffic, let's impose a limit on the number of cars that can drive on the roads each day.

Every car has a unique license plate that consists of 6 alphanumeric characters (repeated letters/digits are possible), where each license plate is guaranteed to have at least 1 number. For example "ABC123", "4QZQZX", and "938103" are all valid license plates. Each day, the government will publish two different digits (0 to 9). If the rightmost digit of a car's license plate matches one of these numbers, then that car is *not* allowed to drive that day. For example, if today's published numbers are 3 and 5, then "ABC123" and "15XTAA" are not allowed to drive, but "4QZQZX" and "35CI98" are.

Assuming that all license plates are unique and of this form (i.e. 6 alphanumeric characters with at least 1 number), what is the maximum number of cars that can be on the road on any given day?

Solution on next page.

Solution

$\frac{8}{10}(36^6 - 26^6)$

The simplest way to arrive at this solution is to count the number of possible license plates, then multiply by $\frac{8}{10}$. 36^6 is the number of unique alphanumeric license plates of length 6, and 26^6 is the number of unique license plates that are only letters (no numbers). So there are $(36^6 - 26^6)$ possible legal license plates. If all legal license plates exist, then exactly $\frac{8}{10}$ of them have the rightmost digit matching the two published numbers (for every license plate format, the rightmost digit can be one of 10 numbers, 8 of which are allowed on any given day; the license plate formats are mutually exclusive), so we simply multiply the total number of legal license plates by $\frac{8}{10}$.

Alternatively, we can consider the (mutually exclusive) sets of license plates that have 1, 2, 3, 4, 5, or 6 digits, and add them all up. If we consider all license plates that have exactly k digits, then there are $\binom{6}{k}10^k 26^{6-k}$ possible license plates of that form (i.e. first choose the locations of k digits among the 6 characters, then select the values of the k digits, then select the identities of the $6 - k$ letters). However, if the rightmost digit cannot be one of the two published numbers, then when we pick the k digits, the rightmost one only has 8 possibilities (instead of 10). Thus, for license plates with k digits, only $\binom{6}{k}10^{k-1}(8)26^{6-k}$ of these can appear on the road on any given day.

Adding up the license plates that can have anywhere from $k = 1$ to $k = 6$ digits, there are at most $\sum_{k=1}^{6} \binom{6}{k}10^{k-1}(8)26^{6-k}$ cars that can be on the road on any given day. By tweaking this expression and invoking Binomial Theorem, we can see that this is the same expression as the solution above:

$\sum_{k=1}^{6} \binom{6}{k}10^{k-1}(8)26^{6-k}$

$= \frac{8}{10}(\sum_{k=1}^{6} \binom{6}{k}10^k 26^{6-k})$ (factor out 8 and $\frac{1}{10}$ from the sum)

$= \frac{8}{10}(\sum_{k=0}^{6} \binom{6}{k}10^k 26^{6-k} - 26^6)$ (make the summation start from $k = 0$ and subtract out the $\binom{6}{0}10^0 26^{6-0}$)

$= \frac{8}{10}((10 + 26)^6 - 26^6)$ (apply Binomial Theorem: $\sum_{k=0}^{n} \binom{n}{k}x^k y^{n-k} = (x + y)^n$)

$= \frac{8}{10}(36^6 - 26^6)$