Solutions to CS 70 Challenge Problems: Basic Probability

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1 General Probability

(a) If 150 fair coins are tossed in the air, what is the probability that exactly 40 of them land up heads?
$\frac{\binom{150}{40}}{2^{150}}$
Suppose we flipped the 150 coins and arranged their results in a line. That light would look something like: HTTHHT...THTHTT. This is our sample space Ω. The sample points we care about have exactly 40 H's, and 110 T's. The number of ways this can happen is $\binom{150}{40}$—of the 150 coins, we choose 40 of them to be heads. The total size of Ω, or the number of all possible outcomes, is $|\Omega| = 2^{150}$—each of the 150 coins can be heads or tails (2 possibilities). Dividing the size of our event by the size of the sample space gives our desired probability.

(b) Recall the Monty Hall problem. Consider a variant of the game where the contestant is faced with 5 doors. He/she selects a single door, and the host reveals the contents of one of the 4 other doors (which turns out to not have the prize). What is the probability the contestant will win the prize if he/she stays with the current door? What is the probability of winning of he/she switches? What is the best strategy?
$P(\text{win if switch}) = \frac{4}{15}$
$P(\text{win if stay}) = \frac{1}{5} = \frac{3}{15}$
Best strategy is to switch.
The reasoning is just like the original Monty Hall problem. Initially, once the contestant has selected a door, we know there is a $\frac{1}{5}$ chance the prize is behind that door, and a $\frac{4}{5}$ chance the prize is behind one of the other 4 doors. Once the host opens one of the other 4 doors, the probability of $\frac{4}{5}$ has been consolidated or coalesced into only 3 doors. If the contestant switches and randomly selects one of those 3 doors, he/she has a $\frac{1}{3} \times \frac{4}{5}$ chance of getting the prize. If he/she stays, it's just the $\frac{1}{5}$ chance. Thus, the best strategy is to switch, even if there are more doors. It turns out that this will be always true, assuming there is only 1 prize.

(c) Carol and Dave each have an identical bag of 5 marbles. Each bag contains a red, orange, yellow, green, blue, and purple marble. Carol reaches into her bag and randomly picks out a marble and puts it into Dave's bag. Dave reaches into his back and randomly picks out a marble and puts it into Carol's bag. What is the probability that after these 2 actions, the bags each have 5 distinctly colored marbles again?
$\frac{1}{3}$
Without loss of generality, let the marble that Carol selected by purple. Then there are 2 purple balls in Dave's bag, out of 6 total. The probability he selects a purple ball to put back into Carol's bag is $\frac{1}{3}$. Notice that the initial color of Carol's marble didn't really matter. We don't have to care about each possibility of what Carol picked, because no matter what Carol picked, it led to the same situation for Dave: to end up with 5 distinct marbles in each bag, Dave would have to have selected one of the 2 marbles of the same color out of 6.
Another way to do this calculation would be to do it for each color. The probability that Carol selected a red marble and Dave put it back is $\frac{1}{5} \times \frac{1}{3}$. We get the same probability for every possible color, and since there are 5 colors (and the events are mutually exclusive), we add them together to get back our original answer of $\frac{1}{3}$.

(d) *Challenge* Alice selects 3 random numbers from the set $\{1, ..., 9\}$, and places them in descending order to form a 3-digit number. Bob does the same thing, but selects 3 random numbers from the set $\{1, ..., 8\}$. Both Alice and Bob select without replacement. What is the probability that Alice's 3-digit number is larger than Bob's 3-digit number?
$\frac{1}{3} + (\frac{2}{3} \times \frac{55}{168})$

There are two (mutually exclusive) cases: either Alice selects a 9, or she doesn't.
If Alice chooses a 9 as one of her digits, she automatically has a larger number. The chance that she selects a 9 is $\frac{\binom{8}{2}}{\binom{9}{3}}$. This evaluates to $\frac{1}{3}$. How did we get this? The number of ways to select 3 numbers where 9 *must* be one of the same as if we fixed the 9, and selected 2 other numbers from the remaining 8. We divide this by the total number of ways to select 3 numbers from her set.
If Alice does not choose a 9 as one of her digits, then her set is the same as Bob's. Notice the symmetry of the situation. The probability that Alice's number A is larger than Bob's B must be the same as the probability that Bob's number B is larger than Alice's A. There's also a certain probability that their numbers are equivalent ($A = B$). Obviously, since these are the only 3 cases, $P(A > B) + P(A < B) + P(A = B) = 2P(A > B) + P(A = B) = 1$. The probability that they both choose the same numbers is simply the probability that Bob chooses any single set of 3 numbers. We fix Alice's selection to be some set of 3 numbers, and the probability they choose the same thing is simply the probability that Bob chooses those exact same 3 numbers. This is $\frac{1}{\binom{8}{3}}$, which evaluates to $\frac{1}{56}$ (we can fix Alice's selection because we know that Alice must pick *something*). Plugging $\frac{1}{56}$ in for $P(A = B)$, we can solve for $P(A > B)$ (again, assuming Alice didn't pick 9): $P(A > B) = \frac{1-P(A=B)}{2} = \frac{1-\frac{1}{56}}{2} = \frac{55}{112}$.
Since Alice will select a 9 with $\frac{1}{3}$ chance, she must *not* select a 9 with $\frac{2}{3}$ chance. Then the chance that Alice doesn't pick a 9 and still has a larger number than Bob is $\frac{2}{3} \times \frac{55}{168}$. Adding our two probabilities, we get the total probability that Alice ends up with a larger number is $\frac{1}{3} + (\frac{2}{3} \times \frac{55}{168})$.

(e) *Challenge* Consider an unfair pair of dice. The two dice are identical, and each has probabilities of rolling 1, 2, 3, 4, 5, and 6, in a ratio of 1:2:3:4:5:6 (e.g. it is six times more likely to roll a 6 than a 1). Upon rolling these two dice, what is the probability of rolling a total of 7?
$\frac{8}{63}$
We start by finding the actual probabilities for each outcome of a single die. If the probability of rolling a 1 is p, then the probability of rolling a 2 is $2p$, of rolling a 3 is $3p$, etc. Then $p+2p+3p+4p+5p+6p = 21p = 1$. Thus, $p = \frac{1}{21}$. Now we can easily find the probability of any single outcome.
We can draw a table of summing outcomes and joint probabilities of the two dice:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

	1	2	3	4	5	6
1	$\frac{1}{21} \times \frac{1}{21}$	$\frac{1}{21} \times \frac{2}{21}$	$\frac{1}{21} \times \frac{3}{21}$	$\frac{1}{21} \times \frac{4}{21}$	$\frac{1}{21} \times \frac{5}{21}$	$\frac{1}{21} \times \frac{6}{21}$
2	$\frac{2}{21} \times \frac{1}{21}$	$\frac{2}{21} \times \frac{2}{21}$	$\frac{2}{21} \times \frac{3}{21}$	$\frac{2}{21} \times \frac{4}{21}$	$\frac{2}{21} \times \frac{5}{21}$	$\frac{2}{21} \times \frac{6}{21}$
3	$\frac{3}{21} \times \frac{1}{21}$	$\frac{3}{21} \times \frac{2}{21}$	$\frac{3}{21} \times \frac{3}{21}$	$\frac{3}{21} \times \frac{4}{21}$	$\frac{3}{21} \times \frac{5}{21}$	$\frac{3}{21} \times \frac{6}{21}$
4	$\frac{4}{21} \times \frac{1}{21}$	$\frac{4}{21} \times \frac{2}{21}$	$\frac{4}{21} \times \frac{3}{21}$	$\frac{4}{21} \times \frac{4}{21}$	$\frac{4}{21} \times \frac{5}{21}$	$\frac{4}{21} \times \frac{6}{21}$
5	$\frac{5}{21} \times \frac{1}{21}$	$\frac{5}{21} \times \frac{2}{21}$	$\frac{5}{21} \times \frac{3}{21}$	$\frac{5}{21} \times \frac{4}{21}$	$\frac{5}{21} \times \frac{5}{21}$	$\frac{5}{21} \times \frac{6}{21}$
6	$\frac{6}{21} \times \frac{1}{21}$	$\frac{6}{21} \times \frac{2}{21}$	$\frac{6}{21} \times \frac{3}{21}$	$\frac{6}{21} \times \frac{4}{21}$	$\frac{6}{21} \times \frac{5}{21}$	$\frac{6}{21} \times \frac{6}{21}$

The only time when the dice sum to 7 is in a single diagonal of 6 possible combinations. We simply take the corresponding probabilities of those 3 entries and add them up:
$(\frac{1}{21} \times \frac{6}{21}) + (\frac{2}{21} \times \frac{5}{21}) + (\frac{3}{21} \times \frac{4}{21}) + (\frac{4}{21} \times \frac{3}{21}) + (\frac{5}{21} \times \frac{2}{21}) + (\frac{6}{21} \times \frac{1}{21}) = \frac{6+10+12+12+10+6}{441} = \frac{56}{441} = \frac{8}{63}$