Solutions to CS 70 Challenge Problems: Conditional Probability

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1 Conditional Probability

(a) Consider 3 cards. One card is blue on both sides, one card is gold on both sides, and one card is blue on one side and gold on the other side. A single card is randomly and placed on a table. One side of the card is blue. What is the probability the other side of the card is also blue?

$\frac{2}{3}$

We want $P(\text{both sides blue}|\text{we see one side blue})$. By the definition of conditional probability, this is equal to $\frac{P(\text{both sides blue} \cap \text{we see one side blue})}{P(\text{we see one side blue})}$.

Note that the numerator just simplifies to $P(\text{both sides blue})$. This is because the event that both sides are blue is a strict subset of the event that we see one side is blue (so the intersection is simply the smaller set). This probability we know is $P(\text{both sides blue}) = \frac{1}{3}$.

Now the denominator is slightly trickier. We want the probability we see one side is blue. We can calculate this using independence, but we can also use counting: there are really 6 possible outcomes (3 cards to pick from, and for each card, we could be shown 1 of 2 sides). Each of these outcomes has equal probability. There are 3 outcomes that match our event (we see one side blue): 2 outcomes if the card is the double-sided blue card, and 1 outcome if the card is the mixed. That makes it a $\frac{1}{2}$ chance that we see one side blue.

Plugging into our initial equation, we get that $P(\text{both sides blue}|\text{we see one side blue}) = \frac{1/3}{1/2} = \frac{2}{3}$

2 Bayes' Rule

(a) Due to your many hours of diligent studying for the next CS 70 midterm, you have successfully learned 80% of the material. That is, on the exam, each question has an 80% chance of being one which you know the correct answer to. On the questions you don't know the answer to, however, you can still guess. Since all of the questions are multiple choice, you will have a 20% chance of getting a question right if you guess. A week later, after the test is over, you decide to look through the graded exam and see what you still need to review. For any question, given that you got the correct answer, what is the probability that you actually knew the answer (rather than just guessing)?

Hint: It might help to first calculate the probability that you will give the correct answer to any question.

$\frac{20}{21}$

Let K be the event you knew the question, and let C be the event you answered it correctly. We will start by calculating $P(C)$. By the total probability rule, $P(C) = P(C \cap K) + P(C \cap \overline{K})$. Using the definition of conditional probability, that's $P(C) = P(C|K)P(K) + P(C|\overline{K})P(\overline{K})$. These we can just plug in. The probability we get an question correct given we know it is 1, and the probability we know any question is 0.8. The probability we get an answer correct given we don't know it is 0.2 (since we guess). There is also a 0.2 chance that we don't know any given question. Thus, $P(C) = (1 \times 0.8) + (0.2 \times 0.2) = 0.84$.

Now what we really want is $P(K|C)$. By Bayes' Rule, this is $P(K|C) = \frac{P(C|K)P(K)}{P(C)}$. We know $P(C|K)P(K) = 1 \times 0.8 = 0.8$, and we know $P(C) = 0.84$. Thus, $P(K|C) = \frac{0.8}{0.84} = \frac{80}{84} = \frac{20}{21} \approx 0.952$.

(b) A robot is trying to cross the road safely. It has a sensor to help it determine the road's conditions, but because of imperfect hardware and changing conditions, it may not always give correct information. The robot knows that if the conditions are truly safe, the sensor will correctly tell it so with $\frac{5}{6}$ probability. But if the conditions are unsafe, the sensor may still tell the robot that it is safe with $\frac{1}{10}$ probability. The robot also knows that in general, the road is safe to cross about $\frac{1}{3}$ of the time. Given that the robot's sensor indicates that it is safe to cross, what is the probability it actually is safe to cross?

$\frac{75}{93}$
Let S be the event that it is actually safe to cross, and let I be the event that the sensor indicates it is safe to cross. The strategy here is identical to the one in the previous problem. We want $P(S|I)$, which is $P(S|I) = \frac{P(I|S)P(S)}{P(I)}$, from Bayes' Rule.
$P(I) = P(I \cap S) + P(I \cap \overline{S}) = P(I|S)P(S) + P(I|\overline{S})P(\overline{S}) = \frac{5}{6} \times \frac{1}{3} + \frac{1}{10} \times \frac{2}{3} = \frac{5}{18} + \frac{1}{15} = \frac{31}{90}$. Note that we used the fact that $P(\overline{S}) = 1 - P(S)$.
$P(I|S)P(S) = \frac{5}{6} \times \frac{1}{3} = \frac{5}{18}$.
$P(S|I) = \frac{P(I|S)P(S)}{P(I)} = \frac{5/18}{31/90} = \frac{75}{93}$

3 Independence

(a) 2 cards are chosen randomly from a deck of card with replacement (after drawing the first card, it is replaced and the deck is shuffled before drawing the second). Given the first card was a spade, what is the probability that the second card is also a spade?
$\frac{1}{4}$.
Since first card is replaced, the second card is completely independent of the first card. Concretely, knowing what the first card is gives no information on what the second card could be. In general, if A and B are independent, $P(A \cap B) = P(A)P(B)$, and therefore $P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$.

(b) Consider rolling 2 normal 6-sided dice. Let A be the event that the first die comes up as an odd number. Let B be the event that the second die comes up as an odd number. Let C be the event that the sum of the dice values is odd. Intuitively, are A, B, C pairwise independent? That is, are any pair of these events independent? What are the probabilities of each event? Of each pair?
They are all pairwise independent. Knowing the first die coming up odd tells us nothing about the outcome of the second die, and since we don't know anything about the second die, the sum of the values could also be anything (even or odd). A symmetric case holds for the second die. Additionally, knowing the sum of the values being odd doesn't give any information on what each individual die could yield. This can be verified mathematically: $P(A) = \frac{1}{2}$
$P(B) = \frac{1}{2}$
$P(C) = \frac{1}{2}$ This may not be obvious at first. This is true due to the symmetry of each die (each die has an equivalent number of even faces and odd faces). You can also convince yourself of this probability by writing out the 36-entry chart.
$P(A \cap B) = \frac{1}{4}$
$P(B \cap C) = \frac{1}{4}$
$P(A \cap C) = \frac{1}{4}$ (completely symmetric to $P(B \cap C)$)
These last 3 probabilities may not be very obvious. It is easiest to see this from the chart—simply count the number of entries that match the query (e.g. for $P(B \cap C)$, count the number of entries out of 36 that have the second die showing an odd number and the summation being odd).
Note that for each joint probability, it is equivalent to the product of the individual probabilities. For example, $P(A \cap B) = P(A)P(B)$.
For a more air-tight mathematical proof of independence, we would have to repeat this for the complement events, as well.

(c) Continuing from part (b), are the events A, B, C mutually independent? That is, is each one independent from the other two? What is the probability of all 3 of them happening together?
They are not mutually independent. Recall that mutual independence means that $P(A \cap B \cap C) = P(A)P(B)P(C)$. This is not true in this case.
In this case, once we know the results of any two of the events, we immediately know the result of the third. For example, if we know both A and B happened (i.e. both dice turned up odd), then they must sum to an even number, so C cannot happen. Similarly, if we know A and C happened, then it must be that B cannot have happened.
Mathematically, $P(A \cap B \cap C) = 0$, because they can never happen together. Clearly, this is not equal to the product of their individual probabilities.