# CS 70 Challenge Problems: Countability and Counting

Solutions at https://alextseng.net/teaching/cs70/

Alex Tseng

## 1 Finiteness, Countable Infinity, Uncountable Infinity

Classify the following sets as finite, countably infinite, or uncountably infinite. Give a short justification.

(a) Set of all prime numbers.

(b) Set of all functions from $\{0,1\}$ to $\mathbb{N}$.

(c) Set of all functions from $\mathbb{N}$ to $\{0,1\}$.

(d) Set of all possible colors that can be encoded by standard HTML RGB (Each color is a 3-tuple of values from `00` to `FF`).

(e) Set of all possible colors that we can see outside in the natural world.

## 2 Countability

(a) Prove that the Cartesian product of any (finite) number of countable sets is countable.

(b) Consider a perfectly balanced binary tree of infinite depth. How many leaves are there?

(c) Consider a square with side length 1. Are there more points inside the square than on one side of the square? Formally justify your response.

## 3 Counting

(a) How many non-decreasing sequences of $k$ numbers are there if all the numbers are drawn (repetition allowed) from the set $\{1, ..., n\}$? For example, one such sequence is $\{1, 3, 3, 6, 9\}$ if $n = 9$ and $k = 5$.

(b) How many ways are there to put $n$ distinct keys on a keyring?

(c) How many ways are there to put $n$ distinct keys on a keyring, where exactly two of those keys cannot be right next to each other? Assume $n \geq 4$

(d) *Challenge* How many ways are there to arrange $n$ elements, where $k$ of those elements can't be adjacent to each other? For example, for $n = 9, k = 3$, this is the number of anagrams of "COMPUTERS", where no two vowels are adjacent. You may assume this is always possible ($k \leq \lceil \frac{n}{2} \rceil$).