CS 70 Challenge Problems:
Graphs, Trees, and Hypercubes

Solutions at https://alextseng.net/teaching/cs70/
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1 Trees

(a) Prove or give a counterexample: a tree must have some vertex of degree 1.

2 Hypercubes

(a) We define the following function:
$s(0) = 1$
$s(n) = 2^{n-1}(s(n-1))^2$
Prove that an n-dimensional hypercube has at least $s(n)$ spanning trees.
Note that a spanning tree of graph G is a subgraph that is a tree, and that connects all vertices of G.

(b) Is the bound in the question above a tight bound? That is, is it possible that an n-dimensional hypercube has more than $s(n)$ spanning trees?

3 General Graphs

(a) Let G be an undirected graph with $n \geq 2$ vertices. Every vertex in G has an even degree, except for the vertices u and v, which have odd degrees. Prove or give a counterexample: there must be a path between u and v.

(b) For some graph G, the graph $\overline{G}$ (G-complement) has all of the same vertices, but has the opposite edges. That is, if an edge (u, v) exists in G, then it does not exist in $\overline{G}$, and if the edge does not exist in G, then it does exist in $\overline{G}$. Prove that for any graph G, either G is connected or $\overline{G}$ is connected. You may assume that the graph is undirected.

(c) *Challenge* Prove that a complete undirected graph on n vertices (assume n is even) can be partitioned into $\frac{n}{2}$ edge-disjoint spanning trees. That is, we can find $\frac{n}{2}$ different spanning trees where no edge is ever used twice.

(d) *Challenge* Given a tournament H of n vertices, show that there exists some node that is reachable from every other node on a path that is at most length 2. That is, we can identify some vertex x in H where if we start on any other vertex in H, we can travel to x by traversing at most 2 edges.
Recall that a tournament is a directed graph where every pair of nodes u and v are connected by either an edge (u, v) or (v, u), but not both.