Solutions to CS 70 Challenge Problems: Stable Marriage

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1 Stable Marriage Properties

(a) Recall that the optimal man for a woman is her most preferred man that she may be paired with, across all possible stable pairings. True or False: two women may have the same optimal man.
False. Recall that the Stable Marriage Algorithm is male optimal, so it pairs each man with his optimal woman. If we run the algorithm with the women proposing to the men, then we topologically swap men and women, producing a female optimal pairing. This pairing guarantees that every woman is paired with her optimal man. Since this pairing is valid and stable, it also guarantees that each woman is paired with someone different. Then no two women can have the same optimal man.

(b) True or False: In executing the Stable Marriage Algorithm, if the algorithm lasts n days, then there exists a woman who is not proposed to on day $n-3$, assuming $n>3$.
True. This is a bit of a trick question. If the algorithm lasts n days, then there exists a woman who is not proposed to until the very last day. This is because the algorithm continues onto a new day whenever there are unpaired men who are not on a string (men who were rejected). Once every woman is proposed to, every man is on a string, and thus the algorithm terminates. A more concise statement would be: there exists a woman who is not proposed to until day n.

(c) True or False: Upon running the Stable Marriage Algorithm, it is possible that every man gets his last choice.
False. Assume for contradiction that there is some run of the algorithm where every man gets his last choice. Then each man must have been rejected by every women except his last choice. If there are n men/women, then that's $n(n-1)$ rejections total. Since each woman can only reject up to $n-1$ men (a woman cannot reject every possible man), each woman must make $n-1$ rejections. From the previous part, we know that there is a woman who is proposed to for the first time on the last day of the algorithm. This woman cannot possibly reject anyone. This contradicts our finding that each woman makes $n-1$ rejections. Therefore it is impossible that every man gets his last choice.

(d) For a normal run of the Stable Marriage Algorithm with n men and n women, let $P_i(M)$ be the rank of the woman that M proposes to on day i (His first choice is rank 1, last choice is rank n). Let $R_i(W)$ be the number of the men that W has rejected so far up to i, not including any rejections on day i. For any day i, what is the value of $\sum_M P_i(M) - \sum_W R_i(W)$?
We make the crucial observation that $P_i(M)$ is equal to one more than the number of times he has been rejected up to day i, but not including day i. Every time he is rejected, he proposes to his next preferred woman, and thus $P_i(M)$ increases by 1, as well. Then if the total number of rejections up to day i but not including it is T, then $\sum_M P_i(M) = T+n$. As for $R_i(W)$, $R_i(W)$ is simply the number of rejections that woman W has given out up to day i, so $\sum_W R_i(W) = T$. Therefore $\sum_M P_i(M) - \sum_W R_i(W) = T+n-T = n$

(e) Consider an instance of the Stable Marriage Problem with n men and n women. In this instance, there are exactly three stable pairings possible: M_1, M_2, M_3. Every woman is matched to a different man in each of the three matchings, so every woman has a clear ranking of which matching she would prefer (according to her preference list). It turns out that some woman W prefers M_3 to M_2 and M_2 to M_1 ($M_3 > M_2 > M_1$). True or False: every woman must have the same ranking for the matchings $M_3 > M_2 > M_1$.
Remember to make a distinction between a stable pairing and the Stable Marriage Algorithm. The algorithm outputs a specific pairing, where there can be multiple stable pairings. In this problem, it is crucial to understand this distinction.

We run the Stable Marriage Algorithm to produce a male optimal and female pessimal ordering. Let this pairing be P. In this pairing, every woman is paired with the worst man she could be paired with across all stable pairings. Now we run the algorithm again with women proposing. This produces another stable pairing O that is male pessimal and woman optimal. In this pairing, every woman is paired with the best man she could be paired with across all stable pairings. We are given that there are three possible pairings and that each woman is paired with a different man in each of the three pairings. Then we know that every woman will most prefer the pairing O because every woman is paired with her optimal man in O. Similarly, she will least prefer the pairing P because every woman is paired with her pessimal man in P. The third possible pairing must be somewhere in between, better than pessimal but worse than optimal. Every woman must find this pairing better than P but worse than O. Thus, every woman has the same ranking for the three possible matchings.